

مذكرة رقم 6 في درس الحساب المثلثي (ملخص)

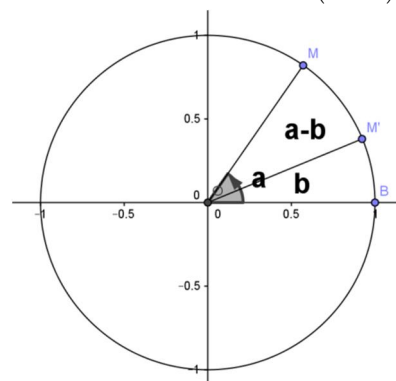
الأهداف و القدرات المنتظرة من الدرس :

محتوى البرنامج	القدرات المنتظرة	توجيهات تربوية
- صيغ التحويل؛ - تحويل الصيغة $a \cos x + b \sin x$	- التمكن من مختلف صيغ التحويل؛ - التمكن من حل معادلات ومتراجحات مثلثية تؤول في حلها إلى المعادلات والمتراجحات الأساسية؛ - التمكن من تمثيل وقراءة حلول معادلة أو متراجحة مثلثية على الدائرة المثلثية.	- ينبغي توخي البساطة في تقديم هذا الفصل وذلك باستعمال أي تقنية في متناول التلاميذ؛ - يتم توظيف الدائرة المثلثية لحل متراجحات مثلثية بسيطة على مجال من IR.

I. صيغ التحويل

(C) دائرة مثلثية مركزها O

معلم متعامد ممنظم $(0; \vec{i}; \vec{j})$



a أفصول منحنى للقطعة OM و b أفصول منحنى للقطعة M'O
 $OM(\cos a; \sin a)$ و $OM'(\cos b; \sin b)$

$$\textcircled{1} \overline{OM} \cdot \overline{OM}' = \cos a \cos b + \sin a \sin b$$

$$\textcircled{2} \overline{OM} \cdot \overline{OM}' = \|\overline{OM}\| \|\overline{OM}'\| \cos(a-b) = \cos(a-b)$$

من : $\textcircled{1}$ و $\textcircled{2}$ نستنتج : $\cos(a-b) = \cos a \cos b + \sin a \sin b$
يمكن أن نبين أيضا أن :

$$\textcircled{2} \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\textcircled{3} \sin(a+b) = \sin a \cos b + \sin b \cos a$$

$$\textcircled{4} \sin(a-b) = \sin a \cos b - \sin b \cos a$$

مثال: أحسب $\sin \frac{\pi}{12}$ و $\cos \frac{\pi}{12}$

$$\text{أجوبة: } \cos \frac{\pi}{12} = \cos \left(\frac{4\pi - 3\pi}{12} \right) = \cos \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$$

$$\cos \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

يمكننا استعمال نتائج الجدول التالي:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

$$\cos \frac{\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3}$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

تابع صيغ أخرى:

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$$

نقسم البسط والمقام على $\cos a \cos b$ فنجد:

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b}}{\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b}} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$$

$$\textcircled{5} \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\textcircled{6} \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

مثال: أحسب $\tan \frac{\pi}{12}$

$$\text{الجواب: } \tan \frac{\pi}{12} = \tan \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$\tan \frac{\pi}{12} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - 1^2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

تمرين 1:

1. أحسب $\tan \frac{5\pi}{12}$ و $\sin \frac{5\pi}{12}$ و $\cos \frac{5\pi}{12}$

2. أحسب $\tan \frac{7\pi}{12}$ و $\sin \frac{7\pi}{12}$ و $\cos \frac{7\pi}{12}$

3. بين أن : $\cos x = \cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right)$

أجوبة: $\cos \frac{5\pi}{12} = \cos \left(\frac{2\pi + 3\pi}{12} \right) = \cos \left(\frac{2\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{6} + \frac{\pi}{4} \right)$

$$\cos \left(\frac{\pi}{6} + \frac{\pi}{4} \right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$$

$$\cos \frac{5\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{6} + \frac{\pi}{4} \right) = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \frac{\pi}{6}$$

$$\sin \frac{5\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

نعلم أن: $\sin^2 a + \cos^2 a = 1$ يعني $\cos^2 a + \sin^2 a = 1$ يعني $\sin^2 a = 1 - \cos^2 a$ يعني $\sin^2 a = 1 - \left(\frac{1}{2}\right)^2$

يعني $\sin^2 a = \frac{3}{4}$ يعني $\sin a = \frac{\sqrt{3}}{2}$ أو $\sin a = -\frac{\sqrt{3}}{2}$ ونعلم أن: $0 < a < \frac{\pi}{2}$

اذن: $\sin a = \frac{\sqrt{3}}{2}$

نعلم أن: $\sin(a+b) = \sin a \cos b + \sin b \cos a$

اذن: $\sin(a+b) = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$

II. نتائج صيغ التحويل و صيغ أخرى

$\cos(2a) = 1 - 2\sin^2 a$ و $\cos(2a) = \cos^2 a - \sin^2 a$

اذن: $\cos^2 a = \frac{1 + \cos 2a}{2}$ و $\cos(2a) = 2\cos^2 a - 1$

$\cos^2 a + \sin^2 a = 1$ و $\sin^2 a = \frac{1 - \cos 2a}{2}$

$1 + \tan^2 a = \frac{1}{\cos^2 a}$ و $\sin(2a) = 2\sin a \cos a$

تمرين 4: علما أن: $\sin x = \frac{1}{3}$ و $x \in \left]0; \frac{\pi}{2}\right[$

أحسب $\sin(2x)$ و $\cos(2x)$

أجوبة: نعلم أن: $\cos^2 x = 1 - \sin^2 x$

اذن: $\cos^2 x = 1 - 2\left(\frac{1}{3}\right)^2 = 1 - \frac{2}{9} = \frac{7}{9}$

و نعلم أن: $\sin(2x) = 2\sin x \cos x$ ومنه يجب حساب $\cos x$:

لدينا: $\cos^2 x = 1 - \sin^2 x$ يعني $\cos^2 x + \sin^2 x = 1$ يعني $\cos^2 x = 1 - \left(\frac{1}{3}\right)^2$

يعني $\cos^2 x = \frac{8}{9}$ يعني $\cos x = \frac{\sqrt{8}}{3}$ أو $\cos x = -\frac{\sqrt{8}}{3}$ ونعلم أن: $x \in \left]0; \frac{\pi}{2}\right[$

اذن: $\cos x = \frac{\sqrt{8}}{3}$ ومنه: $\sin(2x) = 2 \times \frac{1}{3} \times \frac{\sqrt{8}}{3} = \frac{2\sqrt{8}}{9}$

تمرين 5: أحسب $\cos \frac{\pi}{8}$ و $\sin \frac{\pi}{8}$ (لاحظ أن $\frac{\pi}{4} = 2 \times \frac{\pi}{8}$)

أجوبة: حساب: $\cos \frac{\pi}{8}$

نستعمل العلاقة: $\cos^2 a = \frac{1 + \cos 2a}{2}$ ونضع مثلا: $a = \frac{\pi}{8}$

ونجد: $\cos^2 \frac{\pi}{8} = \frac{1 + \cos \frac{\pi}{4}}{2}$ يعني $\cos^2 \frac{\pi}{8} = \frac{1 + \frac{\sqrt{2}}{2}}{2}$ يعني $\cos^2 \frac{\pi}{8} = \frac{2 + \sqrt{2}}{4}$

يعني $\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}}$ أو $\cos \frac{\pi}{8} = -\sqrt{\frac{2 + \sqrt{2}}{4}}$

ولكننا نعلم أن: $0 \leq \frac{\pi}{8} \leq \frac{\pi}{2}$ اذن: $\cos \frac{\pi}{8} \geq 0$ ومنه: $\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}}$

حساب: $\sin \frac{\pi}{8}$: يمكننا استعمال احدي القواعد التالية: $\sin^2 a = \frac{1 - \cos 2a}{2}$ أو

$\sin(2a) = 2\sin a \cos a$ أو $\cos^2 a + \sin^2 a = 1$

لدينا: $\sin^2 a = \frac{1 - \cos 2a}{2}$ ونضع مثلا: $a = \frac{\pi}{8}$

ونجد: $\sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2}$ يعني $\sin^2 \frac{\pi}{8} = \frac{1 - \frac{\sqrt{2}}{2}}{2}$ يعني $\sin^2 \frac{\pi}{8} = \frac{2 - \sqrt{2}}{4}$

يعني $\sin \frac{\pi}{8} = \sqrt{\frac{2 - \sqrt{2}}{4}}$ أو $\sin \frac{\pi}{8} = -\sqrt{\frac{2 - \sqrt{2}}{4}}$

$$\tan \frac{5\pi}{12} = \frac{\sin \frac{5\pi}{12}}{\cos \frac{5\pi}{12}} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{(\sqrt{6} + \sqrt{2})^2}{6 - 2} = \frac{(\sqrt{6} + \sqrt{2})^2}{4}$$

$$\tan \frac{5\pi}{12} = \frac{(\sqrt{6} + \sqrt{2})^2}{4} = \frac{8 + 2\sqrt{12}}{4} = \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}$$

$$\cos \frac{7\pi}{12} = \cos \left(\frac{4\pi + 3\pi}{12} \right) = \cos \left(\frac{4\pi}{12} + \frac{3\pi}{12} \right) = \cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right)$$

$$\cos \left(\frac{\pi}{3} + \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4}$$

$$\cos \frac{7\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\textcircled{4} \sin \left(\frac{\pi}{3} + \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \frac{\pi}{3}$$

$$\sin \frac{7\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\tan \frac{7\pi}{12} = \frac{\sin \frac{7\pi}{12}}{\cos \frac{7\pi}{12}} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{2} - \sqrt{6}}{4}} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}} = \frac{(\sqrt{6} + \sqrt{2})^2}{2 - 6} = \frac{(\sqrt{6} + \sqrt{2})^2}{-4}$$

$$\tan \frac{7\pi}{12} = \frac{8 + 2\sqrt{12}}{-4} = \frac{8 + 4\sqrt{3}}{-4} = -2 - \sqrt{3}$$

$$\textcircled{?} \cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right) = \cos x$$

$$\cos \left(x + \frac{\pi}{3} \right) + \cos \left(x - \frac{\pi}{3} \right) = \cos \frac{\pi}{3} \cos x - \sin \frac{\pi}{3} \sin x + \cos \frac{\pi}{3} \cos x + \sin \frac{\pi}{3} \sin x$$

$$= \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x = 2 \times \frac{1}{2} \cos x = \cos x$$

$$\sin \left(x + \frac{2\pi}{3} \right) + \sin \left(x - \frac{2\pi}{3} \right) + \sin x = 0$$

الجواب: لدينا

$$\sin \left(x + \frac{2\pi}{3} \right) = \sin x \cos \frac{2\pi}{3} + \sin \frac{2\pi}{3} \cos x = \sin x \cos \left(\pi - \frac{\pi}{3} \right) + \sin \left(\pi - \frac{\pi}{3} \right) \cos x$$

$$\sin \left(x + \frac{2\pi}{3} \right) = -\sin x \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cos x$$

$$\sin \left(x - \frac{2\pi}{3} \right) = \sin x \cos \frac{2\pi}{3} - \sin \frac{2\pi}{3} \cos x = \sin x \cos \left(\pi - \frac{\pi}{3} \right) - \sin \left(\pi - \frac{\pi}{3} \right) \cos x$$

$$\sin \left(x - \frac{2\pi}{3} \right) = -\sin x \cos \frac{\pi}{3} - \sin \frac{\pi}{3} \cos x$$

$$\sin \left(x + \frac{2\pi}{3} \right) + \sin \left(x - \frac{2\pi}{3} \right) + \sin x = -2\sin x \cos \frac{\pi}{3} + \sin x = -\sin x + \sin x = 0$$

$$\text{تمرين 3: علما أن: } 0 < a < \frac{\pi}{2} \text{ و } 0 < b < \frac{\pi}{2} \text{ و } \cos a = \sin b = \frac{1}{2}$$

1. أحسب $\sin a$ و $\cos b$

2. أحسب $\sin(a+b)$

أجوبة: (1) حساب $\cos b$

$$\text{نعلم أن: } \cos^2 b + \sin^2 b = 1 \text{ يعني } \cos^2 b = 1 - \sin^2 b \text{ يعني } \cos^2 b = 1 - \left(\frac{1}{2}\right)^2$$

$$\text{يعني } \cos^2 b = \frac{3}{4} \text{ يعني } \cos b = \frac{\sqrt{3}}{2} \text{ أو } \cos b = -\frac{\sqrt{3}}{2} \text{ ونعلم أن: } 0 < b < \frac{\pi}{2}$$

$$\text{اذن: } \cos b = \frac{\sqrt{3}}{2}$$

حساب $\sin a$

ولكننا نعلم أن : $0 \leq \frac{\pi}{8} \leq \frac{\pi}{2}$ إذن : $\sin \frac{\pi}{8} \geq 0$ ومنه : $\sin \frac{\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

تمرين 6: بين أن : $\forall x \in]0; \frac{\pi}{2}[\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

$$\begin{aligned} \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} &= \frac{\sin 3x \cos x - \sin x \cos 3x}{\sin x \cos x} = \frac{\sin(3x-x)}{\sin x \cos x} \\ &= \frac{\sin(3x-x)}{\sin x \cos x} = \frac{\sin 2x}{\sin x \cos x} = \frac{2 \sin x \cos x}{\sin x \cos x} = 2 \end{aligned}$$

تمرين 7: بين أن : $\forall x \in \mathbb{R}$

$$\sin^2 2x - \cos 2x - 1 = -2 \cos^2 x + \cos 2x \quad (1)$$

$$2 \sin^2 x + 12 \cos^2 x = 5 \cos 2x + 7 \quad (2)$$

الجواب:

$$\sin^2 2x - \cos 2x - 1 = (2 \cos x \sin x)^2 - 2 \cos^2 x + 1 - 1 \quad (1)$$

$$4 \cos^2 x \sin^2 x - 2 \cos^2 x = -2 \cos^2 x \cos 2x$$

$$2 \sin^2 x + 12 \cos^2 x = 2 \sin^2 x + 12(1 - \sin^2 x) = -10 \sin^2 x + 12 \quad (2)$$

$$= \frac{-10}{2}(1 - \cos 2x) + 12 = -5(1 - \cos 2x) + 12 = 5 \cos 2x + 7$$

تمرين 8: بين أن : $\forall x \in \mathbb{R}$

$$\sin 3x = \sin x (3 - 4 \sin^2 x) \quad (1)$$

$$\cos 3x = \cos x (4 \cos^2 x - 3) \quad (2)$$

$$c \cos(4x) = 8 \cos^4 x - 8 \cos^2 x + 1 \quad (3)$$

$$\sin(4x) = 4 \sin x (2 \cos^2 x - \cos x) \quad (4)$$

$$\cos^3 x = \frac{1}{4}(3 \cos x + \cos 3x) \quad (5)$$

$$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x \quad (1)$$

$$= 2 \sin x \cos^2 x + (1 - 2 \sin^2 x) \sin x = 2 \sin x (1 - \sin^2 x) + (1 - 2 \sin^2 x) \sin x$$

$$= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x = 3 \sin x - 4 \sin^3 x = \sin x (3 - 4 \sin^2 x)$$

$$\cos 3x = \cos(2x+x) = \cos x \cos 2x - \sin 2x \sin x \quad (2)$$

$$= \cos x (2 \cos^2 x - 1) + \sin x \times 2 \cos x \sin x = 2 \cos^3 x - \cos x - 2 \cos x \sin^2 x$$

$$= 2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x) = 2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x$$

$$= 4 \cos^3 x - 3 \cos x = \cos x (4 \cos^2 x - 3)$$

$$c \cos(4x) = c \cos(2 \times 2x) = 2 \cos^2 2x - 1 = 2(2 \cos^2 x - 1)^2 - 1 \quad (3)$$

$$= 2(4 \cos^4 x - 4 \cos^2 x + 1) - 1 = 8 \cos^4 x - 8 \cos^2 x + 1$$

$$\sin(4x) = \sin(2 \times 2x) = 2 \sin 2x \cos 2x = 2 \times 2 \sin x \cos x (2 \cos^2 x - 1) \quad (4)$$

$$\sin(4x) = 4 \sin x \cos x (2 \cos^2 x - 1) = 4 \sin x (2 \cos^3 x - \cos x)$$

$$\text{؟؟} \cos^3 x = \frac{1}{4}(3 \cos x + \cos 3x) \quad (5)$$

طريقة 1:

$$\frac{1}{4}(3 \cos x + \cos 3x) = \frac{1}{4}(3 \cos x + \cos(x+2x)) = \frac{1}{4}(3 \cos x + \cos x \cos 2x - \sin x \sin 2x)$$

$$= \frac{1}{4}(3 \cos x + \cos x (2 \cos^2 x - 1) - 2 \sin x \sin x \cos x)$$

$$= \frac{1}{4}(2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x + 3 \cos x) = \frac{1}{4}(4 \cos^3 x) = \cos^3 x$$

طريقة 2: نستعمل صيغة تحويل جداء الى مجموع

$$\cos^3 x = \cos^2 x \times \cos x = \frac{1 + \cos 2x}{2} \times \cos x = \frac{1}{2}(\cos x + \cos 2x \times \cos x)$$

$$\cos^3 x = \frac{1}{2}(\cos x + \frac{1}{2}(\cos 3x + \cos x)) = \frac{1}{2} \cos x + \frac{1}{4} \cos 3x + \frac{1}{4} \cos x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$$

$$\text{ومنه : } \cos^3 x = \frac{1}{4}(3 \cos x + \cos 3x)$$

تابع صيغ أخرى:

$$\tan(2a) = \frac{2 \tan a}{1 - \tan^2 a} \text{ وفق شروط محددة}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \text{ و } \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ و } \tan(x) = \frac{2 \tan(\frac{x}{2})}{1 - \tan^2(\frac{x}{2})}$$

$$\sin x = \frac{2 \tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} \text{ و } \cos x = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

بوضع : $t = \tan(\frac{x}{2})$ و لكل x من \mathbb{R} بحيث $x \neq \frac{\pi}{2} + k\pi$

$$\text{و } x \neq \pi + 2k\pi \text{ و لكل } k \in \mathbb{Z}$$

$$\text{لدينا : } \tan x = \frac{2t}{1-t^2} \text{ و } \cos x = \frac{1-t^2}{1+t^2} \text{ و } \sin x = \frac{2t}{1+t^2}$$

مثال: علما أن : $\tan(\frac{x}{2}) = 3$ أحسب : $\sin x$ و $\cos x$

تمرين 9: علما أن : $Q(x) = 1 + \cos x + \cos 2x$ و $P(x) = \sin 2x - \sin x$

بين أن : $Q(x) = \cos x (2 \cos x + 1)$ و أن $P(x) = \sin x (2 \cos x - 1)$
الجواب:

$$Q(x) = 1 + \cos x + \cos 2x = 1 + \cos x + 2 \cos^2 x - 1 = \cos x + 2 \cos^2 x = \cos x (1 + 2 \cos x)$$

$$P(x) = \sin 2x - \sin x = 2 \sin x \cos x - \sin x = \sin x (2 \cos x - 1)$$

III. تحويل جداء الى مجموع:

$$\cos a \cos b = \frac{1}{2} [\cos(a+b) + \cos(a-b)]$$

$$\sin a \sin b = -\frac{1}{2} [\cos(a+b) - \cos(a-b)]$$

$$\sin a \cos b = \frac{1}{2} [\sin(a+b) + \sin(a-b)]$$

$$\cos a \sin b = -\frac{1}{2} [\sin(a+b) - \sin(a-b)]$$

أمثلة: أكتب على شكل مجموع :

$$\cos 4x \times \cos 6x \quad (3) \quad \sin x \times \sin 3x \quad (2) \quad \cos 2x \times \sin 4x \quad (1)$$

أجوبة : (1)

$$\cos 2x \times \sin 4x = \frac{1}{2}(\sin(2x+4x) - \sin(2x-4x)) = \frac{1}{2}(\sin 6x - \sin(-2x))$$

$$= \frac{1}{2}(\sin 6x + \sin 2x) = \frac{1}{2} \sin 6x + \frac{1}{2} \sin 2x$$

$$\sin x \times \sin 3x = \frac{1}{2}(\cos(x+3x) - \cos(x-3x)) = \frac{1}{2}(\cos 4x - \cos(-2x)) \quad (2)$$

$$\sin x \times \sin 3x = \frac{1}{2}(\cos 4x - \cos(2x)) = \frac{1}{2} \cos 4x - \frac{1}{2} \cos(2x)$$

$$\cos 4x \times \cos 6x = \frac{1}{2}(\cos(4x+6x) + \cos(4x-6x)) = \frac{1}{2}(\cos 10x + \cos(-2x)) \quad (3)$$

$$\cos 4x \times \cos 6x = \frac{1}{2} \cos 10x + \frac{1}{2} \cos(2x)$$

IV. تحويل مجموع إلى جداء:

$$\cos p + \cos q = 2 \cos \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\cos p - \cos q = -2 \sin \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

$$\sin p + \sin q = 2 \sin \left(\frac{p+q}{2} \right) \cos \left(\frac{p-q}{2} \right)$$

$$\sin p - \sin q = 2 \cos \left(\frac{p+q}{2} \right) \sin \left(\frac{p-q}{2} \right)$$

مثال: أكتب على شكل جداء: $\sin 2x + \sin 4x$

$$\text{الجواب: } \sin 2x + \sin 4x = 2 \sin \left(\frac{2x+4x}{2} \right) \cos \left(\frac{2x-4x}{2} \right)$$

$$\sin 2x + \sin 4x = 2 \sin 3x \cos(-x) = 2 \sin 3x \cos 2x$$

تمرين 10:

$$1. \text{ بين } \sin \frac{3\pi}{11} + \sin \frac{7\pi}{11} = 2 \sin \left(\frac{5\pi}{11} \right) \cos \left(\frac{2\pi}{11} \right)$$

$$2. \text{ بين } \sin \frac{3\pi}{11} - \sin \frac{7\pi}{11} = -2 \cos \left(\frac{5\pi}{11} \right) \sin \left(\frac{2\pi}{11} \right)$$

$$3. \text{ استنتج أن: } \frac{\sin \frac{3\pi}{11} + \sin \frac{7\pi}{11}}{\sin \frac{3\pi}{11} - \sin \frac{7\pi}{11}} = \frac{\tan \left(\frac{5\pi}{11} \right)}{\tan \left(\frac{2\pi}{11} \right)}$$

$$\text{أجوبة: (1) } \sin \frac{3\pi}{11} + \sin \frac{7\pi}{11} = 2 \sin \left(\frac{3\pi + 7\pi}{2} \right) \cos \left(\frac{3\pi - 7\pi}{2} \right)$$

$$\sin \frac{3\pi}{11} + \sin \frac{7\pi}{11} = 2 \sin \left(\frac{5\pi}{11} \right) \cos \left(-\frac{2\pi}{11} \right) = 2 \sin \frac{5\pi}{11} \cos \frac{2\pi}{11}$$

$$(2) \sin \frac{3\pi}{11} - \sin \frac{7\pi}{11} = 2 \cos \left(\frac{3\pi + 7\pi}{2} \right) \sin \left(\frac{3\pi - 7\pi}{2} \right)$$

$$\sin \frac{3\pi}{11} - \sin \frac{7\pi}{11} = 2 \cos \left(\frac{5\pi}{11} \right) \sin \left(-\frac{2\pi}{11} \right) = -2 \cos \frac{5\pi}{11} \sin \frac{2\pi}{11}$$

$$(3) \frac{\sin \frac{3\pi}{11} + \sin \frac{7\pi}{11}}{\sin \frac{3\pi}{11} - \sin \frac{7\pi}{11}} = \frac{2 \sin \left(\frac{5\pi}{11} \right) \cos \left(\frac{2\pi}{11} \right)}{-2 \cos \left(\frac{5\pi}{11} \right) \sin \left(\frac{2\pi}{11} \right)}$$

$$= \frac{\sin \left(\frac{5\pi}{11} \right) \cos \left(\frac{2\pi}{11} \right)}{\cos \left(\frac{5\pi}{11} \right) \sin \left(\frac{2\pi}{11} \right)} = -\tan \left(\frac{5\pi}{11} \right) \times \frac{1}{\tan \left(\frac{2\pi}{11} \right)} = -\frac{\tan \left(\frac{5\pi}{11} \right)}{\tan \left(\frac{2\pi}{11} \right)}$$

$$\text{تمرين 11: بين أن: } \frac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x} = \tan 3x \times \tan x$$

$$\text{الجواب: } \cos 2x - \cos 4x = -2 \sin \left(\frac{2x+4x}{2} \right) \sin \left(\frac{2x-4x}{2} \right) = 2 \sin(3x) \sin x$$

$$\cos 2x + \cos 4x = 2 \cos \left(\frac{2x+4x}{2} \right) \cos \left(\frac{2x-4x}{2} \right) = 2 \cos 3x \cos x$$

ملاحظة: $\sin(-x) = -\sin x$ و $\cos(-x) = \cos x$

$$\frac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x} = \frac{2 \sin 3x \sin x}{2 \cos 3x \cos x} = \frac{\sin 3x}{\cos 3x} \times \frac{\sin x}{\cos x} = \tan 3x \times \tan x$$

$$\text{تمرين 12: بين أن: } \cos^2 \frac{5x}{2} - \cos^2 \frac{3x}{2} = -\sin 4x \times \sin x$$

$$\text{الجواب: } \cos^2 \frac{5x}{2} - \cos^2 \frac{3x}{2} = \left(\cos \frac{5x}{2} + \cos \frac{3x}{2} \right) \left(\cos \frac{5x}{2} - \cos \frac{3x}{2} \right)$$

$$\cos \frac{5x}{2} + \cos \frac{3x}{2} = 2 \cos \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) = 2 \cos(2x) \cos \left(\frac{x}{2} \right)$$

$$\cos \frac{5x}{2} - \cos \frac{3x}{2} = -2 \sin \left(\frac{5x+3x}{2} \right) \sin \left(\frac{5x-3x}{2} \right) = -2 \sin(2x) \sin \left(\frac{x}{2} \right)$$

$$\text{ومنه: } \cos^2 \frac{5x}{2} - \cos^2 \frac{3x}{2} = 2 \cos(2x) \cos \left(\frac{x}{2} \right) \times -2 \sin(2x) \sin \left(\frac{x}{2} \right)$$

$$= -2 \cos(2x) \times \sin(2x) 2 \cos \left(\frac{x}{2} \right) \sin \left(\frac{x}{2} \right) = -\sin(4x) \sin x$$

تمرين 13: بين أن: $\sin x + \sin 2x + \sin 3x = 2 \sin x \cos x (1 + 2 \cos x)$

$$\text{الجواب: } \sin x + \sin 2x + \sin 3x = \sin 2x + \sin x + \sin 3x = \sin 2x + 2 \sin 2x \cos x$$

$$= \sin 2x + 2 \sin 2x \cos x = \sin 2x (1 + 2 \cos x) = 2 \sin x \cos x (1 + 2 \cos x)$$

V. تحويل الصيغة: $a \cos x + b \sin x$

مثال 1: $\cos x - \sin x$

$$. a = -1 \text{ و } a = 1$$

$$\text{نحسب: } \sqrt{a^2 + b^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\cos x - \sin x = \sqrt{2} \left(\frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \sin x \right) = \sqrt{2} \left(\cos \frac{\pi}{4} \cos x - \sin \frac{\pi}{4} \sin x \right)$$

$$\cos x - \sin x = \sqrt{2} \cos \left(\frac{\pi}{4} + x \right)$$

مثال 2: حل في $[0, 2\pi]$ المعادلة: $\sqrt{3} \cos x + \sin x = \sqrt{3}$

الجواب: نحول أولا: $\sqrt{3} \cos x + \sin x$

$$. a = 1 \text{ و } a = \sqrt{3}$$

$$\text{نحسب: } \sqrt{a^2 + b^2} = \sqrt{3^2 + 1^2} = \sqrt{4} = 2$$

$$\sqrt{3} \cos x + \sin x = 2 \left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x \right) = 2 \left(\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x \right)$$

$$\sqrt{3} \cos x + \sin x = 2 \cos \left(x - \frac{\pi}{6} \right)$$

$$2 \cos \left(x - \frac{\pi}{6} \right) = \sqrt{3} \Leftrightarrow \sqrt{3} \cos x + \sin x = \sqrt{3}$$

$$\cos \left(x - \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} = \cos \left(\frac{\pi}{6} \right) \Leftrightarrow 2 \cos \left(x - \frac{\pi}{6} \right) = \sqrt{3}$$

$$\text{يعني: } x - \frac{\pi}{6} = -\frac{\pi}{6} + 2k\pi \text{ أو } x - \frac{\pi}{6} = \frac{\pi}{6} + 2k\pi$$

$$\text{يعني: } x = 2k\pi \text{ أو } x = \frac{\pi}{3} + 2k\pi$$

$$\text{ومنه: } S = \left\{ 0; \frac{\pi}{3}; 2\pi \right\}$$